

MINIMAL GRAPHS WITH MICRO-OSCILLATIONS

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ABSTRACT. We show that there are minimal graphs in \mathbb{R}^{n+1} whose intersection with the portion of the horizontal hyperplane contained in the unit ball has arbitrarily large $(n-1)$ -measure. The proof hinges on the construction of minimal graphs that are almost flat but have small oscillations of prescribed geometry.

1. INTRODUCTION

Area bounds for minimal graphs play a key role in the theory of minimal surfaces. In this direction, an outstanding open problem is to ascertain whether the area of a minimal graph on the unit ball is uniformly bounded.

More precisely, let \mathbb{B}^n denote the unit ball of \mathbb{R}^n and let u be a function satisfying the equation

$$(1) \quad \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0$$

in \mathbb{B}^n . This is equivalent to saying that the graph of u ,

$$\Sigma_u := \{(x, u(x)) : x \in \mathbb{B}^n\}$$

is a minimal hypersurface of \mathbb{R}^{n+1} . The question of whether the area of a minimal graph on \mathbb{B}^n is uniformly bounded amounts to asking if there is a constant C , depending only on the dimension, such that

$$(2) \quad \mathcal{H}^n(\Sigma_u) < C$$

for any u as above, where \mathcal{H}^n denotes the n -dimensional Hausdorff measure.

In general, it is not known whether the estimate (2) holds. Of course, since Σ_u is a global minimizer of the area functional among the surfaces with fixed boundary, if u is bounded on \mathbb{B}^n it is clear that the area $\mathcal{H}^n(\Sigma_u)$ is bounded by a non-uniform constant that depends on the oscillation of u , $\max_{\mathbb{B}^n} u - \min_{\mathbb{B}^n} u$. However, uniform estimates for the oscillation of u on the ball are not available, so one could conjecture that the uniform estimate (2) should not hold.

Our objective in this note is to explore the failure of a higher-codimension analog of this bound, which turns out to be considerably simpler. We will be interested in uniform bounds for the $(n-1)$ -dimensional Hausdorff measure of the hypersurface, or rather of its intersection with a hyperplane. To this end, let us denote by Π the portion of the horizontal hyperplane that is contained in the unit ball of \mathbb{R}^{n+1} :

$$\Pi := \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0, |x| < 1\}.$$

Our goal is to show that a codimension 1 analog of the estimate (2) does not hold:

Theorem 1. *The $(n - 1)$ -dimensional measure of the intersection of a minimal graph over the unit n -ball with a hyperplane is not uniformly bounded. Specifically, given any constant c , there is some u satisfying the Equation (1) for which Σ_u and Π intersect transversally but*

$$\mathcal{H}^{n-1}(\Sigma_u \cap \Pi) > c.$$

In fact, Theorem 1 follows from a more general result that we can state as follows:

Theorem 2. *Let S be a compact, connected, properly embedded, orientable hypersurface of \mathbb{B}^n with nonempty boundary. Then, for any integer k and $\epsilon > 0$, there is a minimal graph over the unit n -ball and an open subset $\Pi' \subset \Pi$ such that the intersection $\Sigma_u \cap \overline{\Pi'}$ is given by $\Phi(S)$, where $\Phi : \Pi \rightarrow \Pi$ is a diffeomorphism with $\|\Phi - \text{id}\|_{C^k} < \epsilon$.*

It is clear that Theorem 1 indeed follows from Theorem 2 if one chooses S to be a compact hypersurface of \mathbb{B}^n with area $\mathcal{H}^{n-1}(S) > c$ and ϵ is small enough. Notice that this is a local result, and that the minimal graph Σ_u is not required to be complete.

What makes Theorem 1 much easier than the bound (2) is that it can be analyzed in the linear regime of the minimal surface equation. In fact, the strategy that we have used to prove the theorem (see Section 2) is to construct harmonic functions v on the ball that are small in a C^k norm yet their zero set $v^{-1}(0)$ has a large area. The smallness assumption then permits to promote them to solutions of the minimal surface equation through an iterative procedure that does not change much the size of the zero set. Hence the large $(n - 1)$ -measure of the intersection of the minimal graph Σ_u with the hyperplane Π comes from micro-oscillations that do not significantly contribute to the curvature of the minimal hypersurface (which is almost flat).

Notice that this approach does not work for the bound (2) because, although the analogous estimate for harmonic functions fails too, the failure only occurs for harmonic functions not satisfying a smallness assumption, which cannot be promoted to minimal graphs via iteration.

2. PROOF OF THE MAIN THEOREM

In this section we will prove Theorem 2. For this, a well known result of Whitney ensures that, by perturbing S a little if necessary, one can assume that S is analytic. Now let us consider an open extension S' of S (that is, an open, connected, analytic hypersurface S' of \mathbb{B}^n containing S) and let us denote by Ω a small neighborhood of the hypersurface S whose closure is contained in \mathbb{B}^n and such that $\mathbb{R}^n \setminus \Omega$ is connected. Such a choice of S' and Ω is always possible because S is connected and its boundary is nonempty.

An important ingredient in the proof of the main theorem is the construction of a harmonic function on \mathbb{R}^n for which a small deformation of S' is a structurally stable (portion of a) connected component of its zero set:

Lemma 3. *For any $\epsilon > 0$ there is a harmonic function v on \mathbb{R}^n and some $\delta > 0$ such that the zero set $u^{-1}(0)$ of any function u with $\|u - v\|_{C^k(\Omega)} < \delta$ satisfies*

$$u^{-1}(0) \cap \Omega' = \Psi(S'),$$

where Ω' is an open subset of Ω and $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diffeomorphism with $\|\Psi - \text{id}\|_{C^k(\mathbb{R}^n)} < \epsilon$.

Proof. Let us choose an orientation of S' and denote by ν the corresponding unit normal vector. A natural way to define a harmonic function associated with S' and with some control on its zero set and on its gradient is via the following Cauchy problem:

$$(3) \quad \Delta \tilde{v} = 0, \quad \tilde{v}|_{S'} = 0, \quad \frac{\partial \tilde{v}}{\partial \nu} \Big|_{S'} = 1.$$

The Cauchy–Kowaleskaya theorem ensures the existence of a solution \tilde{v} of the above problem in a small neighborhood of S' , which can be taken to be Ω without any loss of generality. Since $\mathbb{R}^n \setminus \Omega$ is connected, the Lax–Malgrange approximation theorem [2] ensures the existence of a harmonic function $v : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\|v - \tilde{v}\|_{C^k(\Omega)} < \delta,$$

where δ is a small quantity to be specified later.

Now let u be close to v in the sense that $\|u - v\|_{C^k(\Omega)} < \delta$. Then

$$\|u - \tilde{v}\|_{C^k(\Omega)} < 2\delta,$$

so, since S is a component of the nodal set of \tilde{v} and the gradient of \tilde{v} does not vanish by (3), Thom’s isotopy theorem [1, Theorem 20.2] implies that, for small enough δ , there is an open subset $\Omega' \subset \Omega$ and a diffeomorphism Ψ of \mathbb{R}^n with $\|\Psi - \text{id}\|_{C^k(\mathbb{R}^n)} < \epsilon$ such that $\Psi(S') = u^{-1}(0) \cap \Omega'$. The lemma then follows. \square

The observation now is that one can construct a solution to the minimal graph equation on the ball whose zero set is a small perturbation of that of the harmonic function constructed in the previous lemma. More precisely, we have the following:

Lemma 4. *Given any $\delta > 0$, there is a function u satisfying the minimal surface equation (1) in \mathbb{B}^n and a positive constant λ such that $\|\lambda u - v\|_{C^k(\mathbb{B}^n)} < \delta$.*

Proof. Assuming that $k \geq 2$ without loss of generality and taking any $\alpha \in (0, 1)$, let us define a function $F : C^{k,\alpha}(\mathbb{B}^n) \rightarrow C^{k-2,\alpha}(\mathbb{B}^n)$ as

$$F(u) := \frac{1}{2} \nabla u \cdot \nabla \log(1 + |\nabla u|^2).$$

Equation (1) is then expressible as

$$(4) \quad \Delta u - F(u) = 0.$$

Let v be the harmonic function on \mathbb{R}^n that we constructed in Lemma 3. Take a small positive constant ϵ that will be fixed later and consider the iterative scheme

$$(5) \quad \begin{aligned} u_0 &:= \gamma v \\ u_{j+1} &:= \gamma v + w_j \end{aligned}$$

where

$$\gamma := \frac{\epsilon}{2\|v\|_{C^{k,\alpha}(\mathbb{B}^n)}}$$

and the function w_j is the unique solution to the boundary value problem

$$(6) \quad \Delta w_j = F(u_j) \quad \text{in } \mathbb{B}^n, \quad w_j = 0 \quad \text{on } \partial \mathbb{B}^n.$$

Our goal is to show that, for small enough ϵ , u_j converges in $C^{k,\alpha}(\mathbb{B}^n)$ to a function u that satisfies the minimal graph equation (4) in \mathbb{B}^n and is close to γv in a suitable sense. To this end, let us start by noticing that, as an application of the maximum principle to the boundary problem (6), the functions w_j must satisfy

$$\|w_j\|_{C^0(\mathbb{B}^n)} \leq C\|F(u_j)\|_{C^0(\mathbb{B}^n)}.$$

Standard elliptic estimates then yield

$$\begin{aligned} \|w_j\|_{C^{k,\alpha}(\mathbb{B}^n)} &\leq C(\|w_j\|_{C^0(\mathbb{B}^n)} + \|F(u_j)\|_{C^{k-2,\alpha}(\mathbb{B}^n)}) \\ &\leq C(\|F(u_j)\|_{C^0(\mathbb{B}^n)} + \|F(u_j)\|_{C^{k-2,\alpha}(\mathbb{B}^n)}) \\ &\leq C\|F(u_j)\|_{C^{k-2,\alpha}(\mathbb{B}^n)}. \end{aligned}$$

On the other hand, if we assume that $\|u_j\|_{C^{k,\alpha}} < \epsilon$, one can exploit the above estimate to infer in Equation (5) that

$$\begin{aligned} \|u_{j+1}\|_{C^{k,\alpha}(\mathbb{B}^n)} &\leq \gamma\|v\|_{C^{k,\alpha}(\mathbb{B}^n)} + \|w_j\|_{C^{k,\alpha}(\mathbb{B}^n)} \\ &\leq \frac{\epsilon}{2} + C\|F(u_j)\|_{C^{k-2,\alpha}(\mathbb{B}^n)} \\ &\leq \frac{\epsilon}{2} + C\|u_j\|_{C^{k,\alpha}(\mathbb{B}^n)}^3 \\ (7) \quad &\leq \frac{\epsilon}{2} + C\epsilon^3 < \epsilon, \end{aligned}$$

so the norm u_{j+1} is less than ϵ too. Here we have used that

$$\|F(w)\|_{C^{k-2,\alpha}(\mathbb{B}^n)} \leq C\|w\|_{C^{k,\alpha}(\mathbb{B}^n)}^3$$

and the fact that $\gamma\|v\|_{C^{k,\alpha}(\mathbb{B}^n)}$ and $C\epsilon^3$ are upper bounded by $\epsilon/2$. Notice, in particular, that since the first function u_0 of the iteration satisfies

$$\|u_0\|_{C^{k,\alpha}(\mathbb{B}^n)} \leq \frac{\epsilon}{2},$$

the induction argument (7) then implies that

$$(8) \quad \|u_j\|_{C^{k,\alpha}(\mathbb{B}^n)} < \epsilon$$

for all j .

To estimate the difference $u_{j+1} - u_j$, let us use the bound (8) to write

$$\begin{aligned} \|F(u_j) - F(u_{j-1})\|_{C^{k-2,\alpha}(\mathbb{B}^n)} &\leq C(\|u_j\|_{C^{k,\alpha}(\mathbb{B}^n)}^2 + \|u_{j-1}\|_{C^{k,\alpha}(\mathbb{B}^n)}^2)\|u_j - u_{j-1}\|_{C^{k,\alpha}(\mathbb{B}^n)} \\ &\leq C\epsilon^2\|u_j - u_{j-1}\|_{C^{k,\alpha}(\mathbb{B}^n)}. \end{aligned}$$

Since

$$\Delta(u_{j+1} - u_j) = F(u_j) - F(u_{j-1}) \quad \text{in } \mathbb{B}^n, \quad u_{j+1} - u_j = 0 \quad \text{on } \partial\mathbb{B}^n,$$

elliptic estimates then yield

$$\begin{aligned} \|u_{j+1} - u_j\|_{C^{k,\alpha}(\mathbb{B}^n)} &\leq C\|F(u_j) - F(u_{j-1})\|_{C^{k-2,\alpha}(\mathbb{B}^n)} \\ (9) \quad &< C\epsilon^2\|u_j - u_{j-1}\|_{C^{k,\alpha}(\mathbb{B}^n)}. \end{aligned}$$

so that taking ϵ small enough for $C\epsilon^2 < 1$, we infer from (8) and (9) that, as $j \rightarrow \infty$, u_j converges in $C^{k,\alpha}(\mathbb{B}^n)$ to some function u with

$$(10) \quad \|u\|_{C^{k,\alpha}(\mathbb{B}^n)} \leq \epsilon.$$

Since the sequence w_j converges to w in $C^{k,\alpha}(\mathbb{B}^n)$, the function u satisfies the equation

$$(11) \quad u = \gamma v + w,$$

where w is the unique solution to the problem

$$\Delta w = F(u) \quad \text{in } \mathbb{B}^n, \quad w = 0 \quad \text{on } \partial\mathbb{B}^n.$$

As v is a harmonic function, one can then take the Laplacian of (11) to show that u is a solution of the minimal graph equation (1) with boundary conditions $u = \gamma v$ on $\partial\mathbb{B}^n$.

Taking $\lambda := 1/\gamma$, one can now use the bound (10), the relation (11) and the definition of γ to check that

$$\begin{aligned} \|\lambda u - v\|_{C^{k,\alpha}(\mathbb{B}^n)} &= \lambda \|u - \gamma v\|_{C^{k,\alpha}(\mathbb{B}^n)} \leq C\lambda \|F(u)\|_{C^{k-2,\alpha}(\mathbb{B}^n)} \\ &\leq C\lambda \|u\|_{C^{k,\alpha}(\mathbb{B}^n)}^3 \leq C\epsilon^2 < \delta \end{aligned}$$

provided that ϵ is sufficiently small. \square

Theorem 2 readily follows from Lemmas 3 and 4.

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